JOURNAL OF COMPUTATIONAL PHYSICS 112, 1–11 (1994)

Chebyshev Pseudospectral Solution of Advection–Diffusion Equations with Mapped Finite Difference Preconditioning

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Received December 20, 1991; revised April 8, 1993

A new Chebyshev pseudo-spectral algorithm with finite difference preconditioning is proposed for the solution of advection–diffusion equations. A mapping technique is introduced which allows good convergence for any Peclet number both for one-dimensional and two-dimensional problems. Numerical results show that first-order Lagrange polynomials are the optimal mapping procedure for the one-dimensional problem and second-order Lagrange polynomials, for the two-dimensional one.

1. INTRODUCTION

The need to solve implicit equations is a basic requirement for spectral algorithms. For both steady problems, whose solution is sought through a time marching computation, and for unsteady calculations, the use of spectral methods may often be feasible only if an implicit or a semi-implicit procedure is introduced. A typical case is the solution of the unsteady Navier–Stokes equations, where an implicit treatment of the diffusive and of some of the advective terms may be required to increase the maximum permissible time step. The present study is confined to linear systems, and we assume that nonlinear ones can be treated by standard linearization techniques. We consider two model equations:

in one-dimension,

$$\frac{du}{dx} - \varepsilon \frac{d^2u}{dx^2} = f(x), \qquad f(x) = \pi^2 \sin(\pi x) + \pi \cos(\pi x); \tag{1}$$

in two dimensions,

$$p(x,y)\frac{\partial u}{\partial x} + q(x,y)\frac{\partial u}{\partial y} - \varepsilon\, \Delta u + r(x,y)u = f(x,y),$$
$$p(x,y) = 3x - y - 1,$$
$$q(x,y) = 1,$$
$$r(x,y) = \tfrac{1}{2},$$
$$f(x,y) = (\varepsilon\pi^2 + r(x,y))\sin(\pi x)\sin(\pi y) + p(x,y)\cos(\pi x) \times \sin(\pi y)\pi + q(x,y)\pi \sin(\pi x)\cos(\pi y), \tag{2}$$

to be solved within the usual Chebyshev domain with homogeneous Dirichlet boundary conditions. Pseudospectral techniques for the discretization of (1) or (2) lead to the linear system of equations,

$$\mathbf{L}U = F, \tag{3}$$

which must be inverted with the minimum computational effort. To grasp the difficulties underlying the solution of system (3) we begin from the one-dimensional case. If Eq. (1) is discretized with a Chebyshev collocation method the $(n-1)\times(n-1)$ matrix $\mathbf{L}$ represents the pseudospectral approximation to the advective–diffuxive operator, obtained by collocation at the points $(x_j = \cos(j\pi/n), \forall j = 1, \ldots, n-1)$ which are the extrema of the nth degree Chebyshev polynomial in $]-1; 1[$. Matrix $\mathbf{L}$ is nonsymmetric, full, which rules out direct inversion for large values of n, and, moreover, its condition number increases at least like $n \times n$ if $\varepsilon \to 0$ [1]. The rapid increase of the condition number with n is the primary cause of the inefficiency of any

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standard iterative method. To alleviate the problem a preconditioning matrix $\mathbf{H}$ is introduced and system (3) rewritten as

$$\mathbf{H}^{-1}\mathbf{L}U = \mathbf{H}^{-1}F, \tag{4}$$

where $\mathbf{H}$ must satisfy some basic properties for the preconditioning to be effective. A first obvious requirement is that it must be possible to invert $\mathbf{H}$ in an inexpensive way. A second requirement is that $\mathbf{H}^{-1}$ must be a good approximation of $\mathbf{L}^{-1}$. If this last requirement is satisfied we can expect the eigenvalues of $\mathbf{H}^{-1}\mathbf{L}$ to be well clustered, i.e., the spectral condition number of $\mathbf{H}^{-1}\mathbf{L}$ to be small and positive definite. When these requirements are satisfied, even a simple iterative method can be expected to perform well. An iterative method widely used in this context [1] is Richardson's, where the solution U of a linear system of equations $\mathbf{L}U = F$ is sought through a series of successive approximations V^n obtained as

$$V^{n+1} = V^n + \omega(F - \mathbf{L}V^n), \tag{5}$$

where ω is a relaxation parameter that has to be chosen to minimize the spectral radius of the iteration matrix $(\mathbf{I} - \omega\mathbf{L})$. A preconditioned version of the same method is

$$V^{n+1} = V^n + \omega\mathbf{H}^{-1}(F - \mathbf{L}V^n), \tag{6}$$

where matrix $\mathbf{H}$ does not have to be inverted since Eq. (6) can be rewritten as

$$\mathbf{H}(V^{n+1} - V^n) = \omega(F - \mathbf{L}V^n). \tag{7}$$

Equation (7) shows the need for the first of the above discussed requirement, for matrix $\mathbf{H}$: the inversion of $\mathbf{H}$ must be cheap enough to make the solution of system (7) less expensive than the computation of $\mathbf{L}V^n$ (usually performed via a fast Chebyshev transform). Preconditioning techniques have been investigated extensively, using both finite difference and finite elements methods to discretize the left-hand side of (7), but most of the existing applications are concerned with spectral solutions of first order or of Helmholtz type partial differential equations (pde) [2–4]. The present authors found a lack of preconditioning techniques suitable for advection–diffusion equations. The aim of this paper is to propose a new type of preconditioning, based on upwind finite difference and on a mapping operator, suited for matrix $\mathbf{L}$, representative of a pseudo-spectral approximation to advective-diffusive operators. The limiting behaviour of standard finite difference preconditioners is reviewed and the modified algorithm for the one-dimensional problem introduced in Section 2, while in Section 3 the two-dimensional problem (Eq. (2)) is discussed. In Section 2 and in Section 3 an exhaustive number of numerical results are presented to validate the algorithm.

2. BASIC ALGORITHM

This section is devoted to the description of the general formulation. The basic idea rests upon an iterative process, where at each step a finite difference problem has to be solved, whose right-hand side is given by a residual evaluated through a Chebyshev pseudo-spectral calculation. The global procedure can be thought of either as a preconditioning, by finite difference corrections, within a pseudo-spectral computation, or as a finite difference procedure with a deferred correction given by the spectral residuals [2]. We begin our analysis by considering Eq. (1). A preconditioner based on second-order central finite differences for the advective operator, δ_{cx}, and for the diffusive term, δ_{xx},

$$\delta_{cx}u_i - \varepsilon\delta_{xx}u_i = f_u \tag{8}$$

leads to the preconditioned system:

$$\mathbf{D}_c^{-1}\mathbf{L}U = \mathbf{D}_c^{-1}F. \tag{9}$$

The eigenvalues of the preconditioned matrix $\mathbf{D}_c^{-1}\mathbf{L}$ for $\varepsilon = 1$ and for $\varepsilon = 0.1$ for a 21- and a 41-node Chebyshev grid are shown in Figs. 1, 2, 3, 4 and in Table I.

The preconditioned eigenvalues are essentially real with very small imaginary parts, well clustered, and their spread is independent from the number of nodes. Although the central difference discretization of the advection term shows very good performances as a preconditioner it also presents the major drawback of losing its own diagonal dominance for low ε values and, more specifically, when the cell Peclet

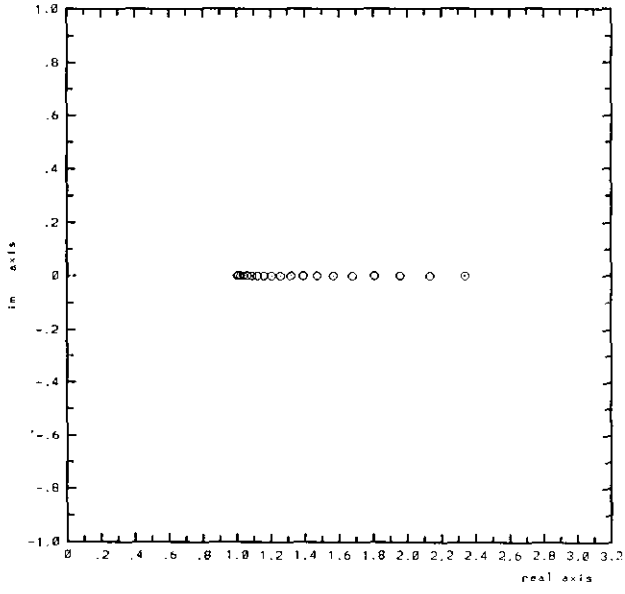

FIG. 1. Eigenvalues distribution of the preconditioned matrix $\mathbf{D}_c^{-1}\mathbf{L}$ at $\varepsilon = 1.0$ for a 21-node collocation.

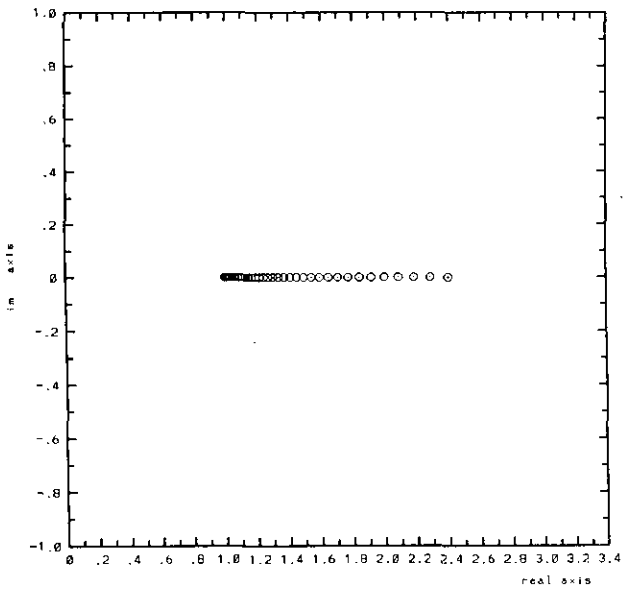

FIG. 2. *Eigenvalues distribution of the preconditioned matrix* $\mathbf{D}_c^{-1}\mathbf{L}$ at $\varepsilon = 1.0$ for a 41-node collocation.

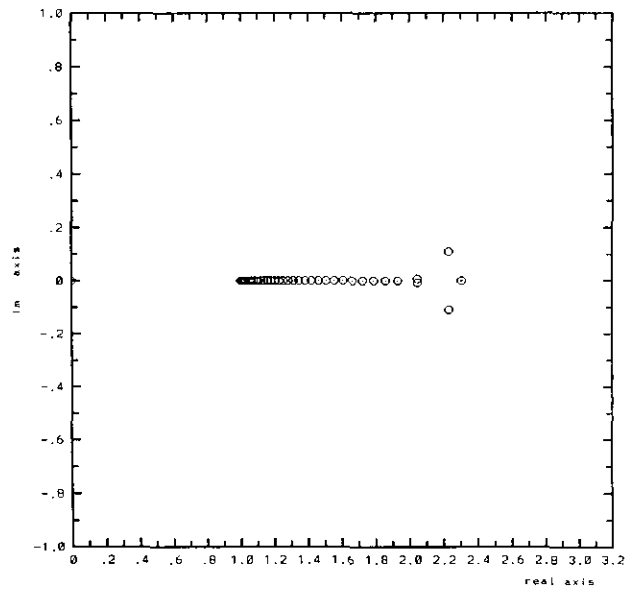

FIG. 4. *Eigenvalues distribution of the preconditioned matrix* $\mathbf{D}_c^{-1}\mathbf{L}$ at $\varepsilon = 0.1$ for a 41-node collocation.

number ($\mathrm{Pe} = 1 \times \Delta x/\varepsilon$, where $\Delta x \sim 1/n$), becomes close to 2 [7]. On the mesh used for the above example it becomes very expensive, if not impossible, to invert the finite difference matrix for value of ε lower than 0.05 since the finite difference matrix itself becomes very ill conditioned. One of the most popular ways to overcome this problem is the introducton of an upwind discretization of the advection term which restores the diagonal dominance of the finite difference matrix. When upwind is introduced in the finite difference preconditioner we obtain the preconditioned linear system of equations

$$\mathbf{D}_{\mathrm{up}}^{-1}\mathbf{L}U = \mathbf{D}_{\mathrm{up}}^{-1}F, \tag{10}$$

where $\mathbf{D}_{\mathrm{up}}^{-1}$ is the inverse of the matrix $\mathbf{D}_{\mathrm{up}}$, built through a second-order upwind discretization of the advective term (δ_x^-) leading to

$$\delta_x^- u_i - \varepsilon\delta_{xx} u_i = f_i. \tag{11}$$

A numerical analysis of the eigenvalues of the preconditioned matrix $\mathbf{D}_{\mathrm{up}}^{-1}\mathbf{L}$ is shown in Figs. 5, 6, 7, 8 and in Table I in which, the first column refers to the preconditioned operator and to the ε value, the second column (the third) is the maximum (minimum) real part of the eigenvalues spectrum, the fourth column contains the maximum imaginary part and the last column yield the spectral condition number $(\lambda_{\max}/\lambda_{\min})$. All the values refer to a 21-point pseudo-spectral Chebyshev approximation.

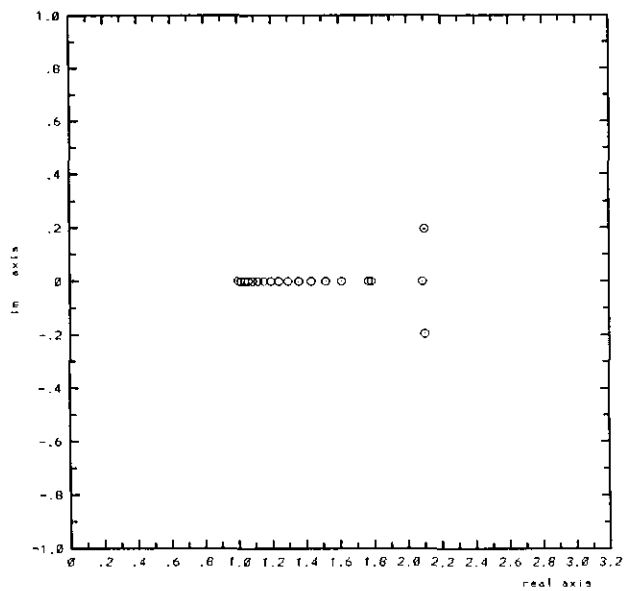

FIG. 3. Eigenvalues distribution of the preconditioned matrix $\mathbf{D}_c^{-1}\mathbf{L}$ at $\varepsilon = 0.1$ for a 21-node collocation.

TABLE I

Testcase	Max $\Re(\lambda)$	Min $\Re(\lambda)$	Max $\Im(\lambda)$	κ
$\mathbf{D}_c^{-1}\mathbf{L}$, $\varepsilon = 1.$	2.3373	1.0000	0.0000	2.3373
$\mathbf{D}_c^{-1}\mathbf{L}$, $\varepsilon = 10^{-1}$	2.1021	1.0000	0.1957	2.1021
$\mathbf{D}_c^{-1}\mathbf{L}$, $\varepsilon = 10^{-1}$	1.2877	0.8809	0.0533	1.4484
$\mathbf{D}_{\mathrm{up}}^{-1}\mathbf{L}$, $\varepsilon = 10^{-2}$	0.9966	0.4102	0.5115	2.4295
$\mathbf{D}_{\mathrm{up}}^{-1}\mathbf{L}$, $\varepsilon = 10^{-3}$	0.9974	0.1671	0.7300	5.9688
$\mathbf{D}_{\mathrm{up}}^{-1}\mathbf{L}$, $\varepsilon = 10^{-4}$	0.9017	−0.0300	0.8222	30.056

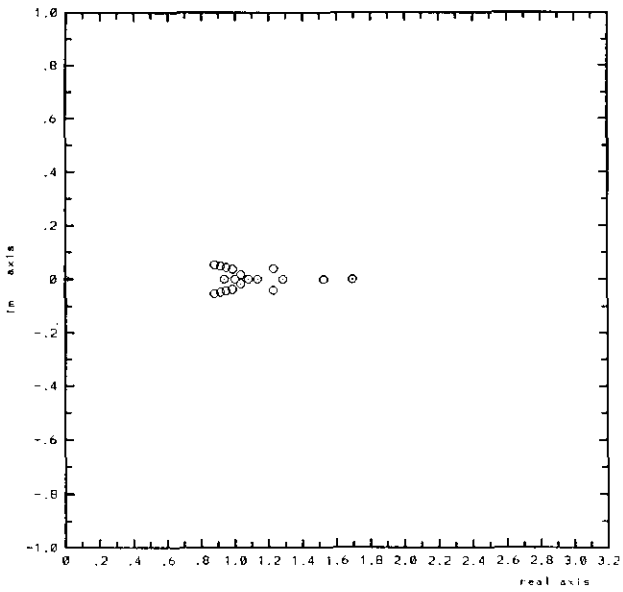

FIG. 5. Eigenvalues distribution of the preconditioned matrix $\mathbf{D}_{up}^{-1}\mathbf{L}$ at $\varepsilon = 0.1$ for a 21-node collocation.

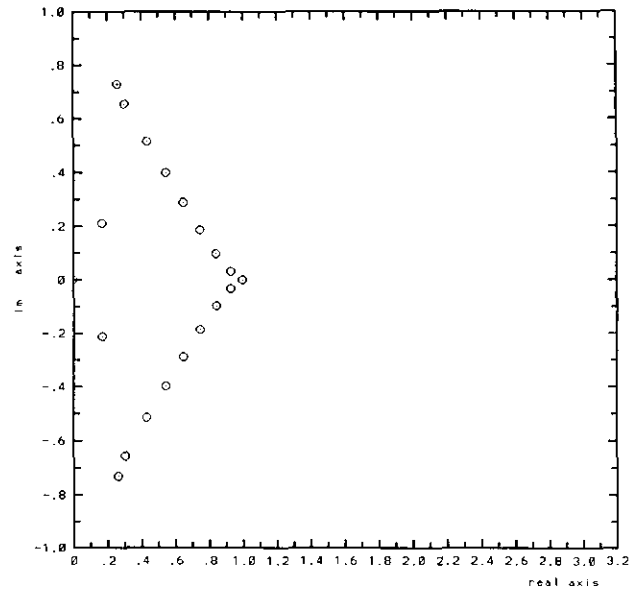

FIG. 7. Eigenvalues distribution of the preconditioned matrix $\mathbf{D}_{up}^{-1}\mathbf{L}$ at $\varepsilon = 0.001$ for a 21-node collocation.

The preconditioned eigenvalues are always complex and show a sharp trend to move toward the negative part of the real axis as ε decreases. It follows that the whole preconditioned matrix $\mathbf{D}_{up}^{-1}\mathbf{L}$ becomes no longer positive definite for low ε values.

We face a situation in which, for low ε, a finite difference central discretization of the advective terms cannot be used to avoid the corruption of the diagonal dominance of the finite difference preconditioner itself, while an upwind discretization of the same terms leads to a nonpositive definite preconditioned matrix. For these reasons we attempt to couple the benefits of the two approaches, namely that upwind finite differences ensure an easy inversion of the preconditioning matrix itself, while central discretization is a more effective preconditioner. This aim can be achieved using a staggered grid preconditioner. The basic idea underlying the staggered grid preconditioning is that of restoring the good behaviour of the central difference precondition-

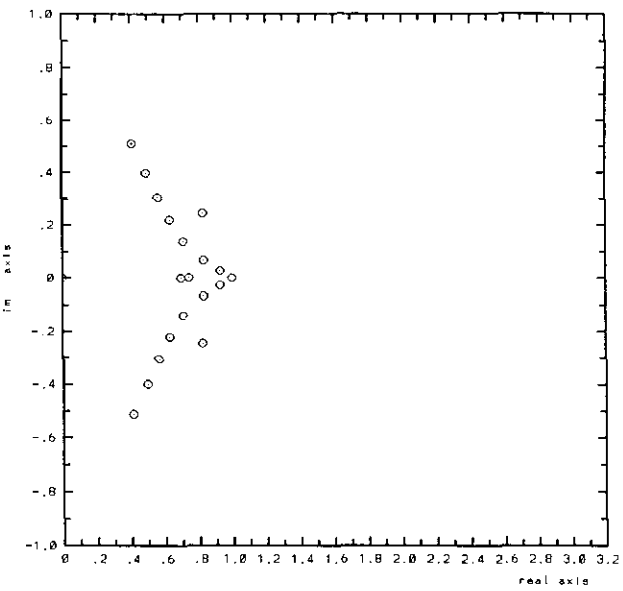

FIG. 6. Eigenvalues distribution of the preconditioned matrix $\mathbf{D}_{up}^{-1}\mathbf{L}$ at $\varepsilon = 0.01$ for a 21-node collocation.

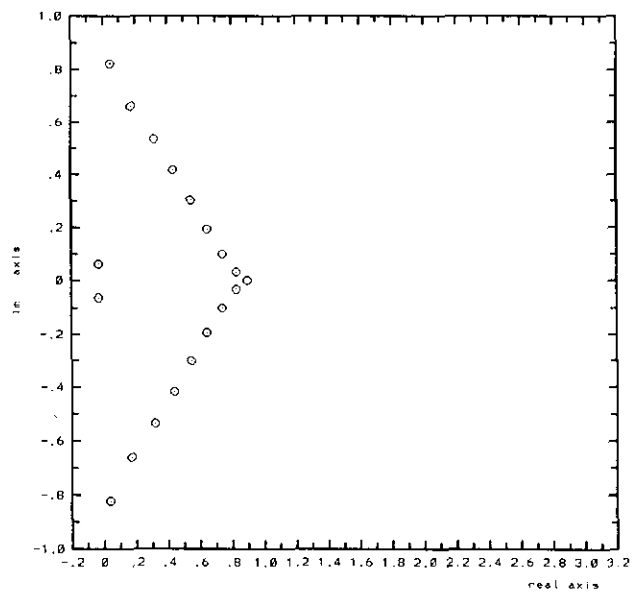

FIG. 8. Eigenvalues distribution of the preconditioned matrix $\mathbf{D}_{up}^{-1}\mathbf{L}$ at $\varepsilon = 0.0001$ for a 21-node collocation.

ing, while using upwind finite differences. To clarify this last point we focus upon the nonperiodic first-order problem

$$\frac{du}{dx} = f(x) \tag{12}$$

with Dirichlet boundary condition at $x = +1$. For this case, Funaro and Rothman [4] proposed the preconditioned system

$$(\mathbf{Z}_n \mathbf{D}_n)^{-1} \mathbf{L}_n U = (\mathbf{Z}_n \mathbf{D}_n)^{-1} F, \tag{13}$$

where $\mathbf{D}_n$ is the first-order upwind finite difference discretization of problem (10) at points $\{x_i\}$ ($x_i = \cos(i\pi/n)$, $\forall i = 0, ..., n$). $\mathbf{L}$ represents the Chebyshev pseudo-spectral approximation to the derivative operator at the collocation points $\{x_i\}$ and $\mathbf{Z}_n$ is a mapping operator between the points $\{x_i\}$ and $\{\xi_i\}$ ($\xi_i = \cos((2i-1)\pi/2n)$, $\forall i = 1, ..., n$). The preconditioned eigenvalues of matrix $(\mathbf{Z}_n \mathbf{D}_n)^{-1} \mathbf{L}$ have been shown [4] to be real and lie between 1 and $\pi/2$. An heuristic understanding of this good behaviour can be found in the fact that the first-order upwind finite difference approximation of the derivative discretized at points $\{x_i\}$ can be regarded as a central one with respect to the set of points $\{\xi_i\}$. This way of thinking can be validated by looking at the preconditioned eigenvalues of the Fourier pseudo-spectral approximation to Eq. (12) with periodic boundary conditions. In this case a finite difference central discretization preconditioner, using the original Fourier nodes would lead to the preconditioned eigenvalues

$$\Lambda_{(p)} = \frac{(p\,\Delta x)}{\sin(p\,\Delta x)}, \tag{14}$$

while the preconditioned eigenvalues for a central discretization on a staggered grid (i.e., the combination of first-order upwind on the original Fourier nodes with a mapping procedure to restore central discretization on the same nodes) are given by

$$\Lambda_{(p)} = \frac{(p\,\Delta x/2)}{\sin(p\,\Delta x/2)}. \tag{15}$$

Eigenvalues (15) can be regarded to be the same as those of a central difference preconditioned system (14), where the Fourier pseudo-spectral approximation to the derivative is collocated at points $\{\psi_i\}$ ($\psi_i = \pi(2i-1)/n$, $\forall i = 1, ..., n$) instead of $\{\phi_i\}$ ($\phi_i = 2\pi i/n$). This suggests that a possible strategy for properly preconditioning the Chebyshev pseudo-spectral approximation to Eq. (1) could be to shift the grid over which the preconditioner is defined. Following the previous discussion we expect an upwind finite difference approximation to the advective–diffusive operator to collocate the solution in a "central fashion" on a particular set of points, which are, a priori unknown. Pursuing this idea we propose to rewrite the finite difference preconditioner to Eq. (1), according to

$$(1-\beta)\,\delta_{cx} u_i + \beta \delta_x^- u_i - \varepsilon(1-\beta)\,\delta_{xx} u_i - \varepsilon\beta \delta_{xx}^- u_i = f_i \tag{16}$$

where δ_x^- is the first-order upwind finite difference operator, while

$$\delta_{xx}^- u_i = \delta_{cx}(\delta_{cx} u_i - \delta_{cx} u_{i-1}). \tag{17}$$

Both the advective and the diffusive finite difference operators collocate their approximations on the same set of points $\{\overline{x}_i\}$, which are related to the corresponding Chebyshev node as

$$\overline{x}_i = x_i - \tfrac{1}{2}\beta(x_i - x_{i-1}). \tag{18}$$

Using an optimal upwind method [5], where β is selected as

$$\beta = \coth\left(\frac{\Delta x_i}{2\varepsilon}\right) - \frac{2\varepsilon}{\Delta x_i} \tag{19}$$

with

$$\Delta x_i = x_i - x_{i-1} \tag{20}$$

on an uniform mesh, a second-order finite difference discretization is obtained for the whole advective–diffusive operator. Formula (18) can also be considered as an exact evaluation of the position, where the scheme is central, for the case of constant advection with respect to x. To prove this point it is sufficient to expand in Taylor's series scheme (16) around a generic $\overline{x}_i$. Since the finite difference approximation of the diffusive term collocates in the same positions as the advective one, we consider only the Taylor's series expansion of

$$(1-\beta)\,\delta_{cx} u_i + \beta \delta_x^- u_i \tag{21}$$

around $\overline{x}_i$. On a uniform mesh we have

$$(1-\beta)\,\delta_{cx} u_i + \beta \delta_x^- u_i = \left(\frac{du}{dx}\right)_{\overline{x}_i} + \frac{1}{2}(\beta h - 2a)\left(\frac{d^2u}{dx^2}\right)_{\overline{x}_i} + \frac{1}{6}(h^2 + 3a^2 - 3a\beta h)\left(\frac{d^3u}{dx^3}\right)_{\overline{x}_i}, \tag{22}$$

where $a = \overline{x}_i - x_i$ and $h = x_{i+1} - x_i = \overline{x_{i+1}} - \overline{x}_i$. The last identity holds only if constant advection is considered.

At the same time a central discretization of du/dx can be written as

$$\frac{\overline{u_{i+1}}-\overline{u_{i-1}}}{\overline{x_{i+1}}-\overline{x_{i-1}}} \simeq \left(\frac{du}{dx_{\overline{x_i}}}+\frac{1}{6}h^2\left(\frac{d^3u}{dx^3}\right)_{\overline{x_i}}\right). \tag{23}$$

By imposing the equivalence of (22) and (23) we obtain two conditions: one on the coefficient of the second-order derivative

$$a=\tfrac{1}{2}\beta h \Rightarrow \bar{x}_i = x_i + \tfrac{1}{2}\beta(x_{i+1}-x_i) \tag{24}$$

and one on the coefficient of the third-order derivative. Formula (24) validates assumption (18). The condition on the third derivative can be used to find the value of β consistent with a second-order discretization and leads to (19) [5, 6]. In case of variable advection, formula (18) can be still considered as a good approximation as long as the advection is represented by a smooth function of x. Summarizing, we expect that a modified finite difference preconditioner, based on discretization (16), collocates in a "central fashion" the solution at the set of points $\{x_i\}$ $(i=0, ..., n)$ defined by the formula (18). Considering a step of a preconditioned Richardson iteration,

$$\mathbf{D}_{\text{mod}}(V^{n+1}-V^n)=\omega(F-\mathbf{L}V^n), \tag{25}$$

if the preconditioner $\mathbf{D}_{\text{mod}}$ is built according to the discretization (16), the left-hand side is collocated in a "central" fashion at points $\{\overline{x_i}\}$ $(i=0, ..., n)$, while on the right-hand side the residuals are taken at the usual Chebyshev nodes. It becomes clear that some mapping operator has to be introduced in Eq. (25). To simplify the notation we use an overbar to indicate the values collocated at the set of points $\{\overline{x_i}\}$. We look for mappings $\mathbf{Z}$ and $\mathbf{W}$ to be applied respectively to the right- and to the left-hand sides of Eq. (25) to make it consistent.Equation (25) can be rewritten as

$$\mathbf{D}_{\text{mod}}\mathbf{Z}(V^{n+1}-V^n)=\omega\overline{\text{RHS}}, \tag{26}$$

where

$$\overline{\text{RHS}}=\mathbf{W}(F-\mathbf{L}V^n). \tag{27}$$

System (26)–(27) can be solved by a two-step procedure, meaning that, first we solve

$$\mathbf{D}_{\text{mod}}(\overline{V}^{n+1}-\overline{V}^n)=\omega\mathbf{W}(F-\mathbf{L}V^n), \tag{28}$$

where the correction $\delta\overline{V^{n+1}}=\overline{V^{n+1}}-\overline{V^n}$ is collocated on the set of points $\{\overline{x_i}\}$, and then we shift the solution $\delta\overline{V^{n+1}}$

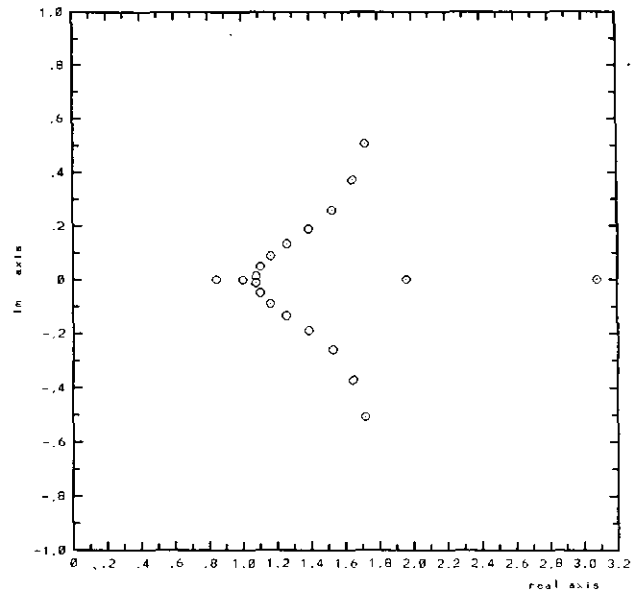

FIG. 9. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{\text{ap}-\text{sp}}\mathbf{D}_{\text{mod}})^{-1}\,\mathbf{WL}$ at $\varepsilon=0.1$ for a 21-node collocation.

on the Chebyshev nodes by inverting the following system of equations:

$$\mathbf{Z}(V^{n+1}-V^n)=(\overline{V^{n+1}}-\overline{V^n}). \tag{29}$$

While the operation involving the mapping $\mathbf{W}$ of the left-hand side of Eq. (28) is just a matrix–vector multiplication, the shift of matrix $\mathbf{Z}$ in Eq. (29) requires a matrix inversion. In order to obtain a matrix similar to $\mathbf{Z}$ but easy to invert, we substitute $\mathbf{Z}$ with some suitable matrix $\mathbf{Z}_{\text{ap}}$ which has to

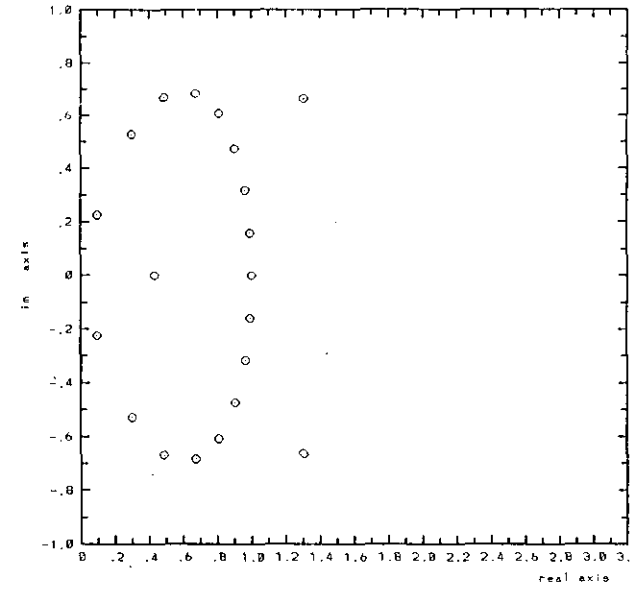

FIG. 10. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{\text{ap}-\text{sp}}\mathbf{D}_{\text{mod}})^{-1}\,\mathbf{WL}$ at $\varepsilon=0.0001$ for a 21-node collocation.

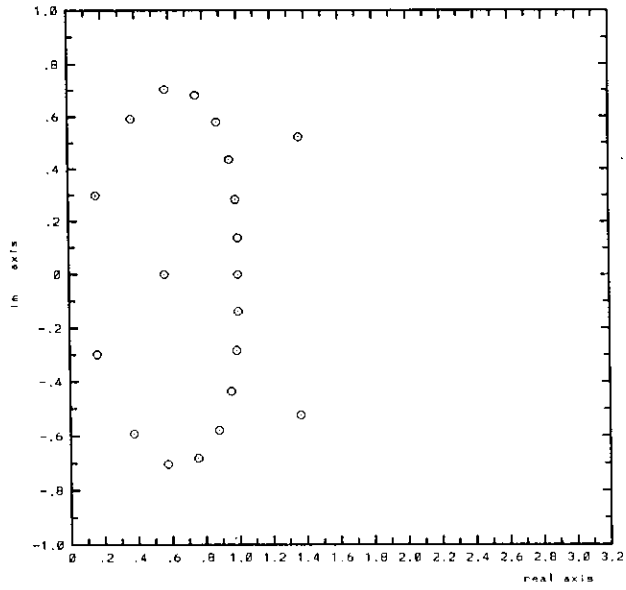

FIG. 11. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{ap-sp}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$ at $\varepsilon = 0.00001$ for a 21-node collocation.

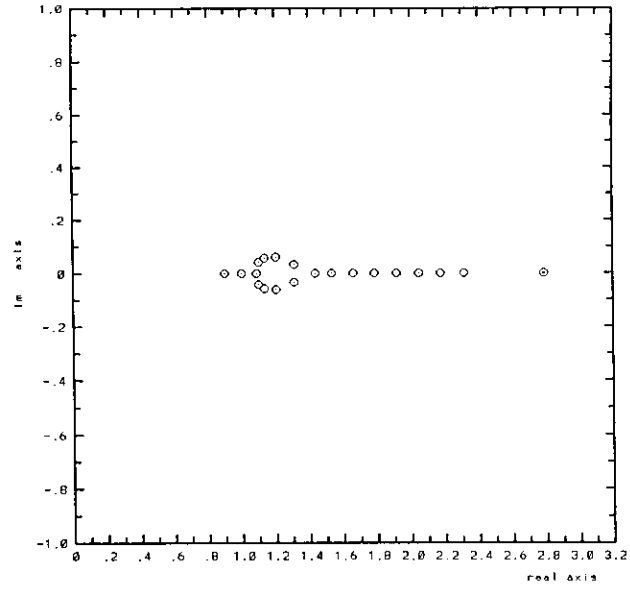

FIG. 12. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{ap-lg2}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$ at $\varepsilon = 0.1$ for a 21-node collocation.

be regarded as an approximation to the operator related to $\mathbf{Z}$. Typically we build the approximate matrix $\mathbf{Z}_{ap}$ by means of polynomials and splines of different degrees which interpolate the values of the corrections from points $\{\overline{x}_i\}$ to the nodes of the Chebyshev grid $\{x_i\}$. In order to obtain a matrix $\mathbf{Z}_{ap}$ that is cheap to invert we consider only interpolations that lead to tridiagonal matrices. Using cubic splines the preconditioned matrix $(\mathbf{Z}_{ap-sp}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$ is found to be positive definite for any value of ε. The positions of the preconditioned eigenvalues at several values of ε for a 21-node Chebyshev pseudo-spectral approximation are shown in Figs. 9, 10, 11 and summarized in Table II.

Although positive defined, the eigenvalues are quite scattered, complex, and, at low ε, some of them have real parts very close to zero. The location of the eigenvalues on the complex plane explains the bad performance of the preconditioned Richardson iterative procedure as shown in Table III. The situation does not improve in a significant way when Lagrangian second-order polynomials are used to build matrix $\mathbf{Z}_{ap}$ (i.e., $\mathbf{Z}_{ap-lg2}$). The eigenvalues of the preconditioned matrix are still complex and some of them have real parts that are very close to zero at low values of ε. Figures 12, 13, and 14 show the locus of the computed eigenvalues of $(\mathbf{Z}_{ap-lg2}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$ for some meaningful (corresponding to high Peclet numbers) values of ε. The same results are summarized in Table II.

No dramatic increase of convergence rate is achieved as shown in Table III. Instead, impressive results are obtained when the mapping is approximated with simple first-order Lagrange polynomials. In this last case the preconditioned eigenvalues are always very clustered and their imaginary parts are quite small; this behavior is found to be independ-

TABLE II

Testcase	Max $\Re(\lambda)$	Min $\Re(\lambda)$	Max $\Im(\lambda)$	κ
$(\mathbf{Z}_{ap-sp}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$, $\varepsilon = 10^{-5}$	1.3043	0.0971	0.6835	13.4325
$(\mathbf{Z}_{ap-lg2}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$, $\varepsilon = 10^{-5}$	1.7615	0.4317	0.4389	4.0803
$(\mathbf{Z}_{ap-lg1}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$, $\varepsilon = 10^{-5}$	1.5199	0.7538	0.3481	2.0163

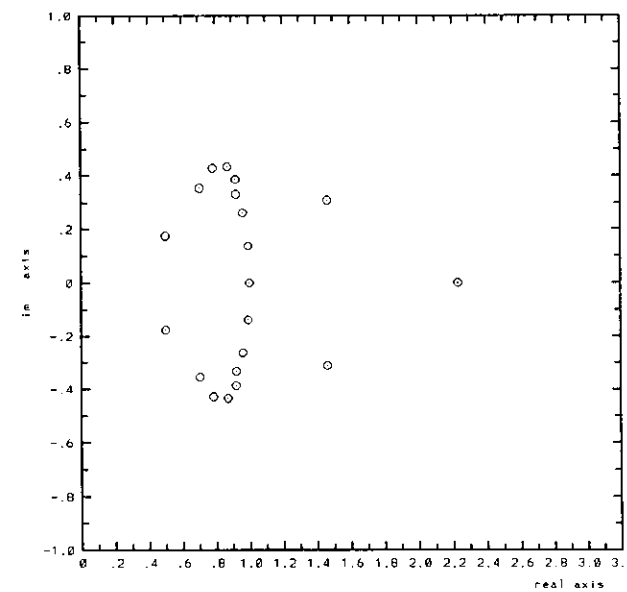

FIG. 13. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{ap-lg2}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$ at $\varepsilon = 0.001$ for a 21-node collocation.

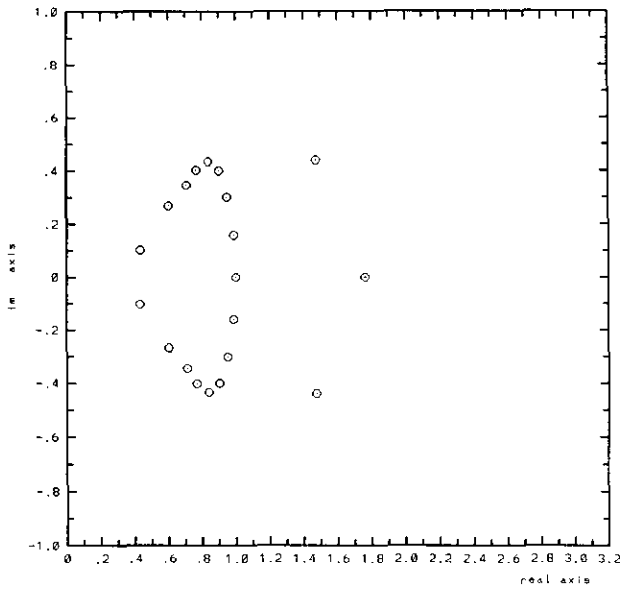

FIG. 14. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{ap-lg2}\mathbf{D}_{mod})^{-1}\mathbf{WL}$ at $\varepsilon = 0.00001$ for a 21-node collocation.

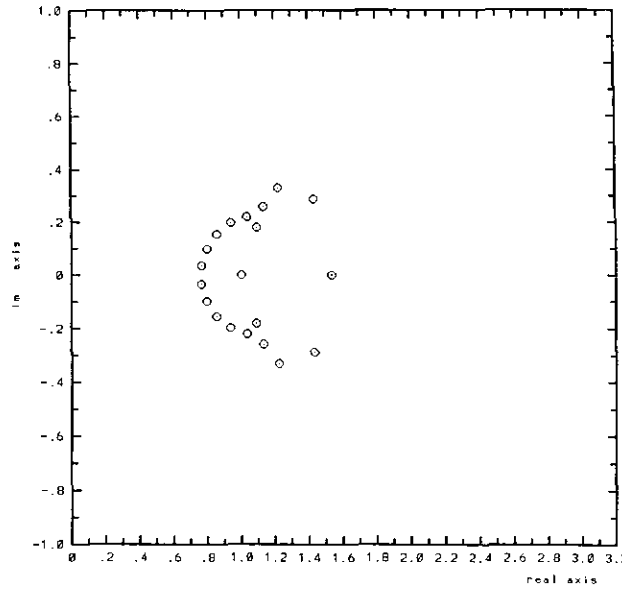

FIG. 16. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{ap-lg1}\mathbf{D}_{mod})^{-1}\mathbf{WL}$ at $\varepsilon = 0.0001$ for a 21-node collocation.

ent from the number of used nodes. Figures 15–18, together with Table II summarize the results.

As a benchmark of the effectiveness of the preconditioning the convergence rate of the already introduced preconditioned Richardson (7) iterative scheme is measured. The iterative process is always started from the finite difference solution of the same differential problem (Eq. (1)), and the optimal relaxation parameter ω is chosen for each computed case according to the relationship $\omega = 2/(\lambda_{min} + \lambda_{max})$. The error between the exact solution of Eq. (1),

$$u(x) = \sin(\pi x), \tag{30}$$

and the approximate solution, as determined by the L2[−1, +1] norm is in the order of 1×10^{-13}. The results of the proposed mapping strategies are compared in Table III.

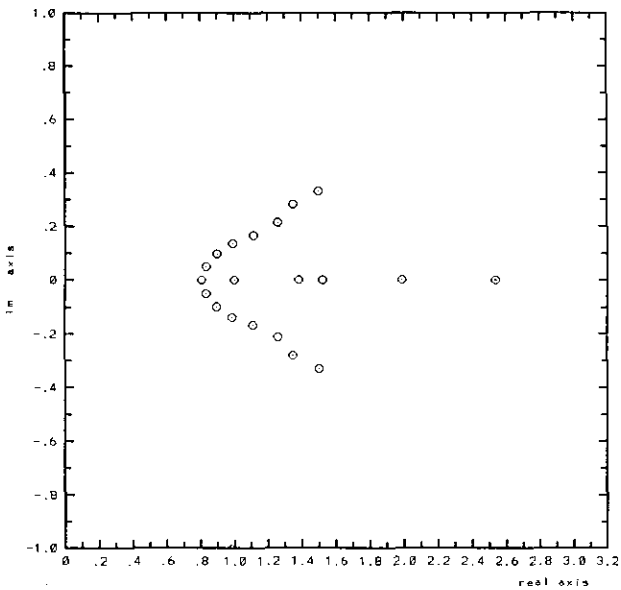

FIG. 15. Eigenvalues distribution of the preconditioned matrix$(\mathbf{Z}_{ap-lg1}\mathbf{D}_{mod})^{-1}\mathbf{WL}$ at $\varepsilon = 0.01$ for a 21-node collocation.

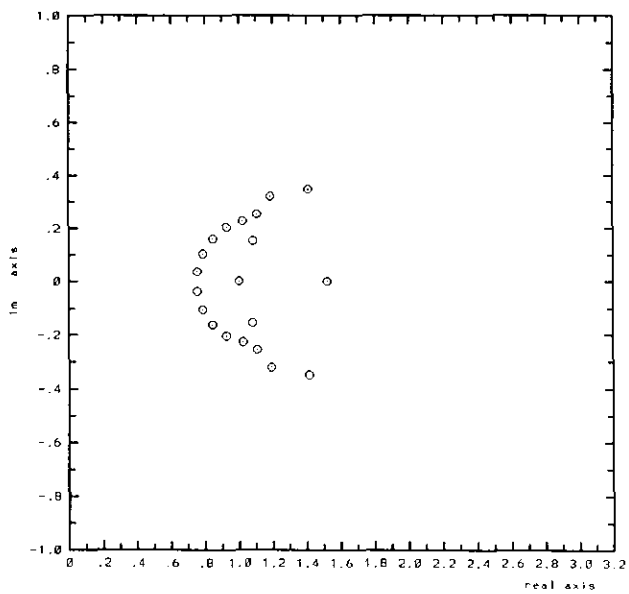

FIG. 17. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{ap-lg1}\mathbf{D}_{mod})^{-1}\mathbf{WL}$ at $\varepsilon = 0.00001$ for a 21-node collocation.

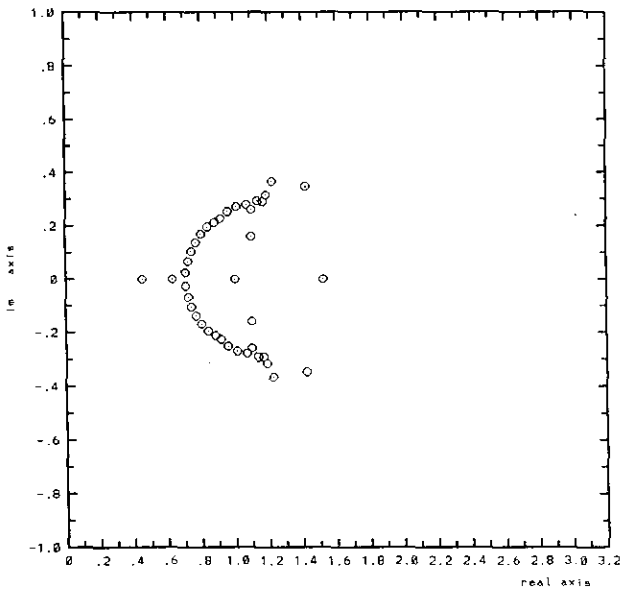

FIG. 18. *Eigenvalues distribution of the preconditioned matrix* $(\mathbf{Z}_{ap-lg1}\,\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$ *at* $\varepsilon = 0.00001$ *for a 41-node collocation.*

3. ALGORITHM EXTENSION TO THE 2D CASE

For Eq. (2), in two-dimensions, the Chebyshev advection–diffusive operator

$$p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y} - \varepsilon\Delta \tag{31}$$

with homogeneous Dirichlet boundary conditions at $\{-1\}\times[-1, 1]$, $\{1\}\times[-1, 1]$, $[-1, 1]\times\{-1\}$, and $[-1, 1]\times\{1\}$ can be represented by the $n^2\times n^2$ matrix $\mathbf{L}^{(2)}$. The Chebyshev pseudo-spectral approximation to Eq. (2) becomes

$$\mathbf{L}^{(2)}(U) = F. \tag{32}$$

A natural extension of the algorithm presented in the previous section implies the introduction of a finite difference discretization to Eq. (2) equivalent to the one applied to Eq. (1). We propose the following finite difference approximation, where the superscript $\mp$ indicates the direction of upwinding, i.e., forward or backward differencing,

$$\begin{aligned}&\overline{p_{i,j}}(1-\beta_x)\,\delta_{cx}\overline{u_{ij}} + \overline{q_{i,j}}(1-\beta_y)\,\delta_{cy}\overline{u_{ij}} + \beta_x\delta_x^{\mp}\overline{u_{ij}}\\ &\quad + \beta_y\delta_y^{\mp}\overline{u_{ij}} - \varepsilon(1-\beta_x)\,\delta_{xx}\overline{u_{ij}} - \varepsilon(1-\beta_y)\,\delta_{yy}\overline{u_{ij}}\\ &\quad - \varepsilon\beta_x\delta_{xx}^{\mp}\overline{u_{ij}} - \varepsilon\beta_y\delta_{yy}^{\mp}\overline{u_{ij}} + \overline{r_{i,j}u_{ij}} = \overline{f_{ij}},\end{aligned} \tag{33}$$

where

$$\overline{p_{i,j}} = p(\overline{x}_i, \overline{y}_j) \tag{34}$$

$$\overline{q_{i,j}} = q(\overline{x}_i, \overline{y}_j) \tag{35}$$

$$\overline{r_{i,j}} = r(\overline{x}_i, \overline{y}_j) \tag{36}$$

$$\overline{f_{i,j}} = f(\overline{x}_i, \overline{y}_j). \tag{37}$$

All the functions are evaluated at shifted points $\{(\bar{x}_{ij}, \bar{y}_{ij})\}$, to be defined later. Again, the finite difference discretization can be regarded as an optimal upwind scheme when the two parameters β_x and β_y are chosen as

$$\beta_x = \coth\frac{p(x, y)\,\Delta x}{2\varepsilon} - \frac{2\varepsilon}{p(x, y)\,\Delta x} \tag{38}$$

$$\beta_y = \coth\frac{q(x, y)\,\Delta y}{2\varepsilon} - \frac{2\varepsilon}{q(x, y)\,\Delta y} \tag{39}$$

and we expect this discretization to collocate in a "central" fashion the solution at the set of points $\{(\overline{x}_{ij}, \overline{y}_{ij})\}$ that, following the same idea applied for the one-dimensional case, can be assessed to be

$$\begin{aligned}\bar{x}_{ij} &= x_i + \tfrac{1}{2}\beta_x(x_{i+1} - x_i) \quad \text{if} \quad p(x_i, y_j) < 0\\ \bar{x}_{ij} &= x_i - \tfrac{1}{2}\beta_x(x_i - x_{i-1}) \quad \text{if} \quad p(x_i, y_j) > 0\end{aligned} \tag{40}$$

$$\begin{aligned}\bar{y}_{ij} &= y_j + \tfrac{1}{2}\beta_y(y_{j+1} - y_j) \quad \text{if} \quad q(x_i, y_j) < 0\\ \bar{y}_{ij} &= y_j - \tfrac{1}{2}\beta_y(y_j - y_{j-1}) \quad \text{if} \quad q(x_i, y_j) > 0.\end{aligned} \tag{41}$$

The position of the shifted point for a generic Chebyshev node is sketched in Fig. 19.

In the present case the finite difference operator $\mathbf{D}_{mod}$, based on discretization (33), collocates the "central" solution on a cloud of scattered points $\{(\overline{x}_{ij}, \overline{y}_{ij})\}$ determined by the local Peclet number according to formulas (40)–(41). Again, we solve the linear system (32) with a preconditioned Richardson scheme

$$\mathbf{H}^{(2)}\delta V_{ij}^{n+1} = \omega\overline{\mathrm{RHS}}^n, \tag{42}$$

where $\mathbf{H}^{(2)} = \mathbf{D}_{mod}^{(2)}\mathbf{Z}^{(2)}$ and $\overline{\mathrm{RHS}}^n = \mathbf{W}^{(2)}(F - \mathbf{L}^{(2)}V^n)$. The overbar still represents the collocation at points $(\overline{x}_{ij}, \overline{y}_{ij})$.

TABLE III

Testcase	*Initial L2 error*	*Iterations*
$\mathbf{D}_{up}^{-1}\mathbf{L}$, $\varepsilon = 10^{-2}$	4.2×10^{-1}	136
$(\mathbf{Z}_{ap-sp}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$, $\varepsilon = 10^{-2}$	6.1×10^{-2}	19
$(\mathbf{Z}_{ap-sp}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$, $\varepsilon = 10^{-3}$	4.3×10^{-2}	122
$(\mathbf{Z}_{ap-lg2}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$, $\varepsilon = 10^{-3}$	4.1×10^{-2}	97
$(\mathbf{Z}_{ap-lg1}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$, $\varepsilon = 10^{-3}$	5.4×10^{-2}	24
$(\mathbf{Z}_{ap-lg1}\mathbf{D}_{mod})^{-1}\,\mathbf{WL}$, $\varepsilon = 10^{-4}$	5.7×10^{-2}	32

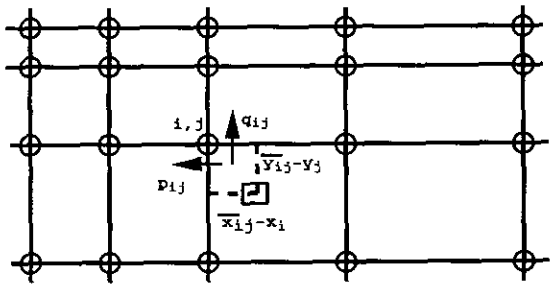

FIG. 19. Sketch of the position of a shifted point for a generic Chebyshev node.

We introduce, as for the one-dimensional case, the operator $\mathbf{Z}_{ap}^{(2)}$ as an approximation to matrix $\mathbf{Z}^{(2)}$ which relates the values computed in a "central fashion" on the staggered grid $\{(\overline{x_{ij}}, \overline{y_{ij}})\}$, to the ones on the two-dimensional Chebyshev grid $\{x_i\}$ and $\{y_j\}$ $(x_i = \cos(i\pi/n),\ y_j = \cos(j\pi/n),\ \forall i, j: i = 0, ..., n; j = 0, ..., n)$. Several approximations to $\mathbf{Z}_{ap}^{(2)}$ have been tested and the best result found when using second-order Lagrange polynomials. For this case we analyse the eigenvalue distribution both for the constant advection and variable advection problems (2). In the first case functions $p(x, y)$ and $q(x, y)$ are selected as constants equal to one; in the second one $p(x, y)$ and $q(x, y)$ are

$$p(x, y) = 3 \times x - y - 1 \tag{43}$$

$$q(x, y) = 1. \tag{44}$$

In the constant advection problem, the eigenvalues of the preconditioned matrix $\mathbf{Z}_{ap}^{(2)^{-1}}\mathbf{D}_{mod}^{(2)^{-1}}\mathbf{W}^{(2)}\mathbf{L}^{(2)}$ are shown in Figs. 20 and 21 for a 11×11 Chebyshev pseudo-spectral collocation at $\varepsilon = 0.001$ and at $\varepsilon = 0.0001$, respectively.

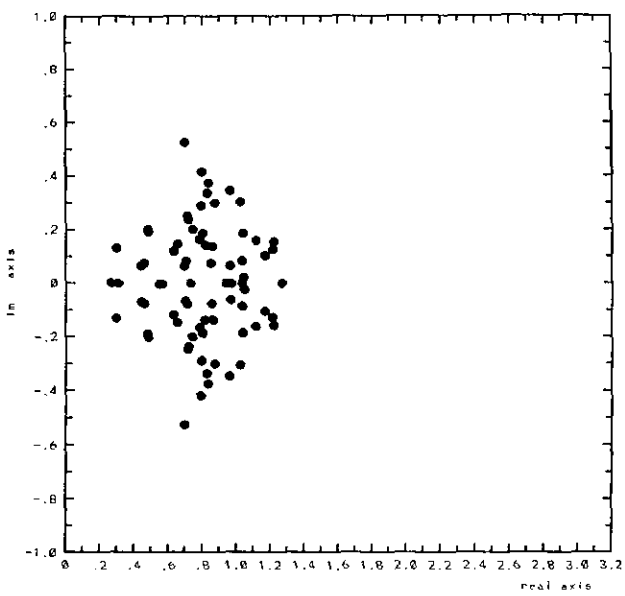

FIG. 21. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{ap}^{(2)}\mathbf{D}_{mod}^{(2)})^{-1}\ \mathbf{W}^{(2)}\mathbf{L}^{(2)}$ in the constant advection case at $\varepsilon = 0.0001$ (bottom) for an 11×11-node collocation.

They are quite clustered and, apparently, safely larger than zero. When a variable advection problem (2) is considered, (18) does not hold anymore. The set of points $\{(\overline{x_{ij}}, \overline{y_{ij}})\}$, determined by formulas (40)–(41) can be considered only as an approximation to the real locations of the shifted points. This consideration may be taken as an

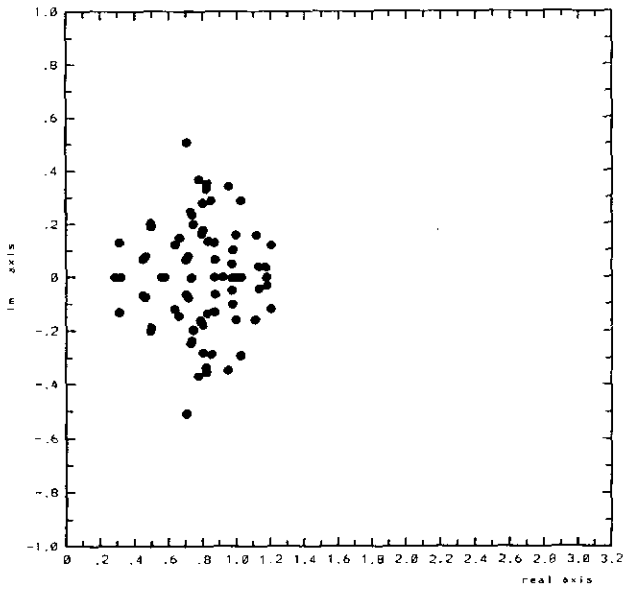

FIG. 20. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{ap}^{(2)}\mathbf{D}_{mod}^{(2)})^{-1}\ \mathbf{W}^{(2)}\mathbf{L}^{(2)}$ in the constant advection case at $\varepsilon = 0.001$ for an 11×11-node collocation.

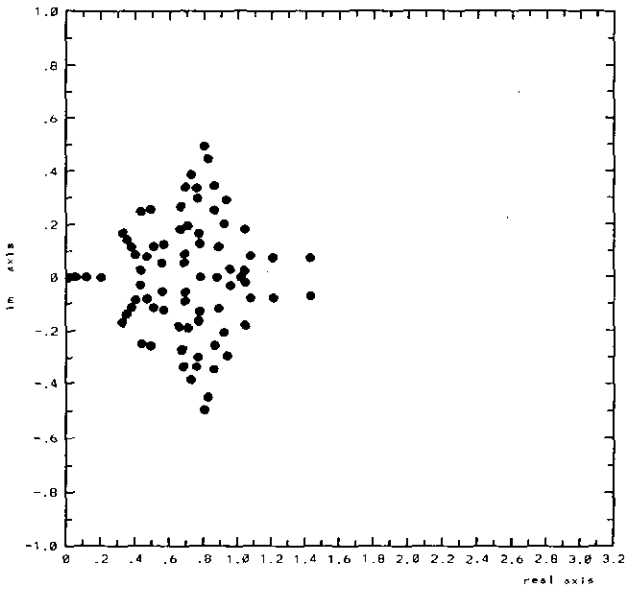

FIG. 22. Eigenvalues distribution of the preconditioned matrix $(\mathbf{Z}_{ap}^{(2)}\mathbf{D}_{mod}^{(2)})^{-1}\ \mathbf{W}^{(2)}\mathbf{L}^{(2)}$ in the variable advection case at $\varepsilon = 0.0001$ for an 11×11-node collocation.

TABLE IV

Testcase	Initial L2 error	Iterations
$(\mathbf{Z}_{ap}^{(2)}\mathbf{D}_{mod}^{(2)})^{-1}\mathbf{W}^{(2)}\mathbf{L}^{(2)}$, $\varepsilon = 10^{-2}$ const adv.	6.8×10^{-1}	37
$(\mathbf{Z}_{ap}^{(2)}\mathbf{D}_{mod}^{(2)})^{-1}\mathbf{W}^{(2)}\mathbf{L}^{(2)}$, $\varepsilon = 10^{-3}$ const adv.	3.3×10^{-1}	112
$(\mathbf{Z}_{ap}^{(2)}\mathbf{D}_{mod}^{(2)})^{-1}\mathbf{W}^{(2)}\mathbf{L}^{(2)}$, $\varepsilon = 1$. nonconst adv.	4.7×10^{-4}	26
$(\mathbf{Z}_{ap}^{(2)}\mathbf{D}_{mod}^{(2)})^{-1}\mathbf{W}^{(2)}\mathbf{L}^{(2)}$, $\varepsilon = 10^{-1}$ nonconst adv.	3.3×10^{-3}	28
$(\mathbf{Z}_{ap}^{(2)}\mathbf{D}_{mod}^{(2)})^{-1}\mathbf{W}^{(2)}\mathbf{L}^{(2)}$, $\varepsilon = 10^{-2}$ nonconst adv.	2.5×10^{-1}	39
$(\mathbf{Z}_{ap}^{(2)}\mathbf{D}_{mod}^{(2)})^{-1}\mathbf{W}^{(2)}\mathbf{L}^{(2)}$, $\varepsilon = 10^{-3}$ nonconst adv.	4.1×10^{-1}	176
$\mathbf{H}_{fe}^{-1}\mathbf{L}^{(2)}$, $\varepsilon = 1$. nonconst adv.	1.0×10^{-2}	14
$\mathbf{H}_{fe}^{-1}\mathbf{L}^{(2)}$, $\varepsilon = 10^{-1}$ nonconst adv.	5.0×10^{-2}	20
$\mathbf{H}_{fe}^{-1}\mathbf{L}^{(2)}$, $\varepsilon = 10^{-2}$ nonconst adv.	7.0×10^{-3}	48
$\mathbf{H}_{fe}^{-1}\mathbf{L}^{(2)}$, $\varepsilon = 10^{-3}$ nonconst. adv.	—	*Unstable*

explanation of the less satisfying (with respect to the previous case) behaviour of the preconditioned eigenvalues at very low values of ε of which an example is given in Fig. 22, for $\varepsilon = 0.0001$ and an 11×11 Chebyshev grid. However, as shown below, full convergence is obtained for all ε. As shown for the one-dimensional case we use the preconditioned Richardson iterative scheme (7) as a benchmark to test the effectiveness of the preconditioner. Again the initial guess for the iteration is the solution of problem (2) obtained by the finite difference approximation (33) or by finite elements (see below). The error between the exact solution of Eq. (2) with variable advection,

$$u(x, y) = \sin(\pi x) \times \sin(\pi y), \tag{45}$$

and the approximate solution, given by the $L2[-1, 1]$ norm is of the order of 1×10^{-13} for a Chebyshev grid of a 21×21 nodes. Table IV summarizes the results for different values of ε and introduces a comparison between the proposed algorithm and the results obtained on the same equation (31) by Deville and Mund [8] with a finite elements preconditioner. It appears that, at relatively high ε values, i.e., $\varepsilon = 0.01$, the proposed finite difference preconditioner, while retaining a quite simple and compact structure, performs as well as the finite elements one. Moreover, it still behaves satisfactorily for advection dominated problems, i.e., ε lower than 0.01, where the finite elements preconditioner encounters severe problems.

4. CONCLUSIONS

It has been shown that neither central nor upwind finite differencing are, in their standard form, well suited as preconditioners for the spectral solution of advection–diffusion equations at high Peclet numbers. Good results can instead be obtained generalizing the approach proposed in [4], namely using a finite difference approximation which is well-conditioned on the spectral grid points $\{x_i\}$ and centered with respect to a shifted grid $\{\xi_i\}$, which is a priori unknown and discretization dependent. Numerical tests have shown that mapping the solution $\{\xi_i\}$ on the grid $\{x_i\}$ leads to a preconditioner which is stable and efficient at all Peclet numbers. The choice of the mapping operator is critical for the efficiency of the method and further analysis is required to determine the optimal form for multidimensional, variable advection problems.

ACKNOWLEDGMENTS

This work was financially supported by the Service à la Programmation de la Politique Scientifique of Belgium (S.P.P.S.) as part of Programme d'Impulsion en Technologie de l'Information—Volet Informatique du Futur (Contract No. IT/IF/21). A. Pinelli is grateful to Daniele Funaro of Universitá di Pavia, Italy, for his precious advice.

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